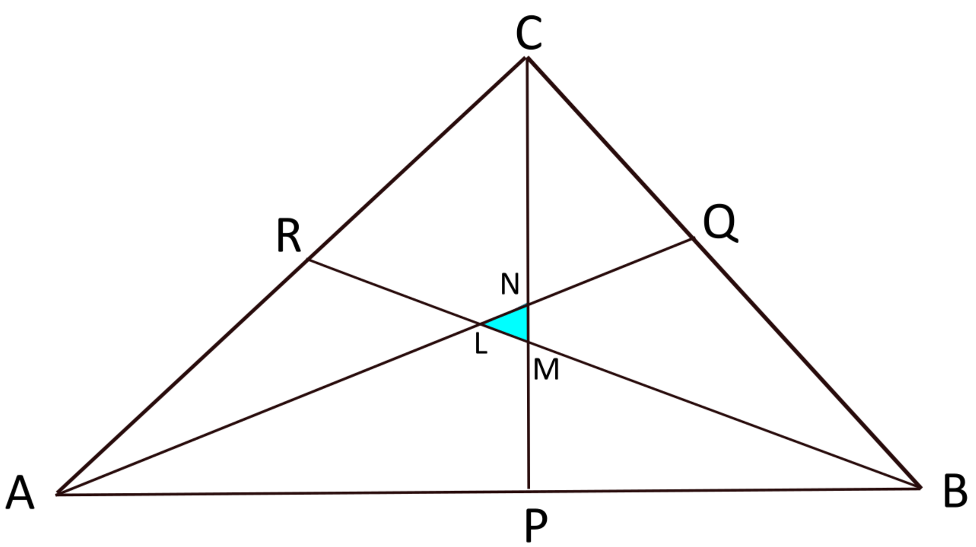
Sample Questions

1. Prove that: $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n + 1)^2$

2. Find the possible solutions of the given equation
$$x^3 - (3 + \sqrt{3})x = 0$$

3. Let *m* and *n* be 2 integers each one not divisible by 3. Use proof by cases to show that *mn* is not divisible by 3.

4. Given AB = 16 cm, BC = 10 cm & AC = 14 cm. RA = RC, AB ⊥ PC, AQ is angle bisector of ∠BAC calculate the area of the shaded triangle LMN. (Take tan $\cos^{-1}(5/\sqrt{28}) = \frac{1}{5}\sqrt{3}$); $\sin[90 - \cos^{-1}((7\sqrt{129})/86)] = 7\sqrt{129})/86$)



Suggested solutions

1. **Step1** Show it is true for **n=1**

$1^3 = ¼ \times 1^2 \times 2^2$ is True

Step 2 Assume it is true for **n=k**

$1^3 + 2^3 + 3^3 + ... + k^3 = ¼k^2(k + 1)^2$ is True (An assumption!)

Step3 Now, prove it is true for "k+1"

$1^3 + 2^3 + 3^3 + ... + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$?

We know that $1^3 + 2^3 + 3^3 + ... + k^3 = ¼k^2(k + 1)^2$ (the assumption above), so we can do a replacement for all but the last term:

$¼k^2(k + 1)^2 + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$

Multiply all terms by 4:

$k^2(k + 1)^2 + 4(k + 1)^3 = (k + 1)^2(k + 2)^2$

All terms have a common factor $(k + 1)^2$, so it can be cancelled:

$k^2 + 4(k + 1) = (k + 2)^2$

And simplify:

$k^2 + 4k + 4 = k^2 + 4k + 4$

They are the same! So it is true.

So: $1^3 + 2^3 + 3^3 + ... + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$ is True

2. $x^3 - \left(3 + \sqrt{3}\right)x = 0$

Let y = $\sqrt{3}$, $y^2 = 3$

$x^3 - (y^2 + y\,)x + y^2 = 0$

$x^3 - (y^2x + yx\,)x + y^2 = 0$

$x^3 - y^2x + yx + y^2 = 0$

$x(x^2 - y^2) - y(x - y) = 0$

$x(x-y)(x+y)-y(x-y)=0$

$(x-y)(x(x+y)-y)=0$

$(x-y)(x^2+xy-y)=0$

$(x-\sqrt{3})(x^2+x\sqrt{3}-\sqrt{3})=0$

$x-\sqrt{3}=0$, $x=\sqrt{3}$

$(x^2+x\sqrt{3}-\sqrt{3})=0$

$using$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-\sqrt{3}\pm\sqrt{\left(\sqrt{3}\right)^2-4(-\sqrt{3})}}{2}$$

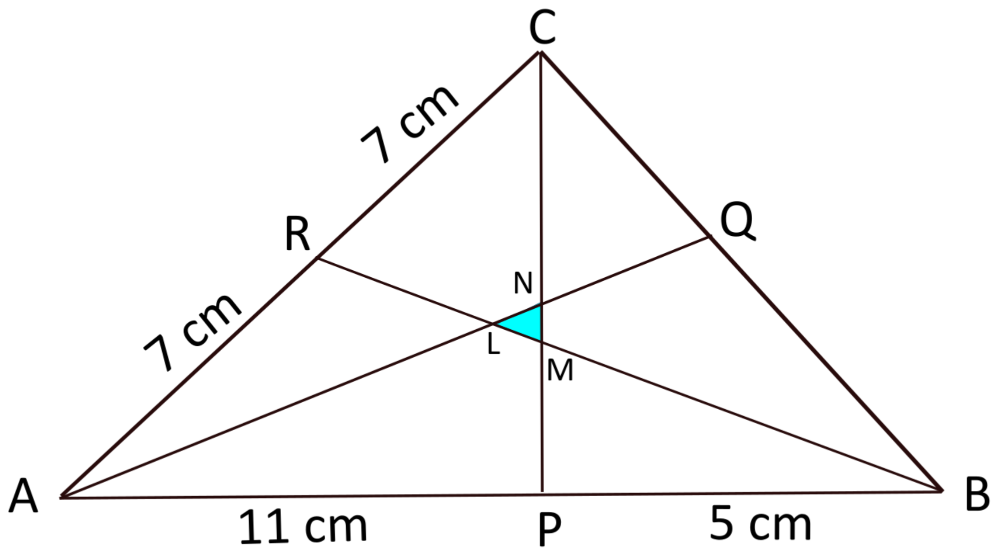
$$x=-\left(\frac{-\sqrt{3}\pm\sqrt{\left(\sqrt{3}\right)^2-4(-\sqrt{3})}}{2}\right)$$

3. There are 3 cases; consider each case in turn.

If $m=3a+1$ and $n=3b+1$, then $mn=(3a+1)(3b+1)=9ab+3a+3b+1$, which is not divisible by 3.

If $m=3a+1$ and $n=3b+2$, then $mn=(3a+1)(3b+2)=9ab+6a+3b+2$, which is not divisible by 3.

If $m=3a+2$ and $n=3b+2$, then $(3a+2)(3b+2)=9ab+6a+6b+4$, which is not divisible by 3.



4. From ΔABC

$AC^2 = AB^2 + BC^2 - 2\times AB\times BC\times\cos(\angle ABC)$ {Law of Cosines}

$\Rightarrow 14^2 = 16^2 + 10^2 - 2\times16\times10\cos(\angle ABC)$

$\Rightarrow 196 = 256 + 100 - 320\cos(\angle ABC)$

$\Rightarrow 320\cos(\angle ABC) = 160$

$\Rightarrow \cos(\angle ABC) = ½$

$\Rightarrow$ **$\angle ABC = 60°$**

From ΔBPC

$\sin(\angle ABC) = PC/BC = PC/10$

$\Rightarrow \sin 60 = PC/10$

$\Rightarrow ½\sqrt{3} = PC/10$

$\Rightarrow$ **$PC = 5\sqrt{3}$ cm**

$\cos(\angle ABC) = PB/BC$

$\Rightarrow \cos 60 = PB/10$

$\Rightarrow ½ = PB/10$

$\Rightarrow$ **PB = 5 cm**

From ΔAPC

AP = AB - PB

$\Rightarrow$ AP = 16 - 5

$\Rightarrow$ **AP = 11 cm**

$\sin(\angle PAC) = = PC/AC$

$\Rightarrow \sin(\angle PAC) = 5\sqrt{3}/11$

cos (∠BAC) = =AP/AC,

⇒ cos (∠BAC) = 11/14

Let ∠QAC = ∠QAP = x {AQ is angle bisector of ∠PAC}
cos (∠BAC) = cos 2x
⇒ cos (∠BAC) = 2 $\cos^2$ x – 1 (double angle identity for cosine)
⇒ 2 $\cos^2$ x - 1 = 11/14
⇒ $\cos^2$ x = 25/28
⇒ cos x = ± 5/√28
0 < x < 180 {x is an angle inside the triangle}
⇒ cos x = 5/√28

⇒ ∠QAP = $\cos^{-1}$ (5/√28)

From ΔARB
$BR^2 = AB^2 + AR^2$ - 2×AB×AR×cos (∠BAR)
⇒ $BR^2 = 16^2 + 7^2$ - 2×16×7×11/14
= 256 + 49 - 176 = 129

⇒ BR = √129 cm
$AR^2 = AB^2 + BR^2$ - 2×AB×BR×cos (∠ABR)
⇒ $7^2 = 16^2$ + 129 - 2×16×√129×cos (∠ABR)
⇒ 49 = 256 + 129 - 32√129 cos (∠ABR)
⇒ cos (∠ABR) = (7√129)/86

⇒ ∠ABR = $\cos^{-1}$ ((7√129)/86)

From ΔAPN
tan ∠QAC = tan $\cos^{-1}$ (5/√28)
⇒ PN/PA = tan $\cos^{-1}$ (5/√28)
⇒ PN/11 = tan $\cos^{-1}$ (5/√28)
= ⅕√3

⇒ PN = ⅕×11√3 cm
∠ANP = 90 -∠QAP
⇒ ∠ANP = 90 - $\cos^{-1}$ (5/√28)
⇒ ∠LNM = 90 - $\cos^{-1}$ (5/√28)
⇒ sin ∠LNM = sin [90 - $\cos^{-1}$ (5/√28)]
= cos $\cos^{-1}$ (5/√28)

⇒ sin ∠LNM = 5/√28

From ΔPBM
tan ∠PMB = PM/PB
⇒ PM/5 = tan ($\cos^{-1}$ ((7√129)/86))

$= 5\sqrt{3}/21$

⇒ **$PM = 25\sqrt{3}/21$ cm**

$\angle PMB = 90 - \angle ABR$

⇒ $\angle PMB = 90 - \cos^{-1}((7\sqrt{129})/86)$

⇒ $\angle LMN = 90 - \cos^{-1}((7\sqrt{129})/86)$

⇒ $\sin \angle LMN = \sin[90 - \cos^{-1}((7\sqrt{129})/86)]$

$= \cos[\cos^{-1}((7\sqrt{129})/86)]$

⇒ **$\sin \angle LMN = 7\sqrt{129})/86$**

$MN = PN - PM$

⇒ $MN = \frac{1}{5}\times 11\sqrt{3} - 25\sqrt{3}/21$

⇒ **$MN = 106\sqrt{3}/105$ cm**

From ΔLMN

$\angle MLN = 180 - (\angle LMN + \angle LNM)$

⇒ $\angle MLN = 180 - [90 - \cos^{-1}((7\sqrt{129})/86) + 90 - \cos^{-1}(5/\sqrt{28})]$

⇒ $\angle MLN = \cos^{-1}((7\sqrt{129})/86) + \cos^{-1}(5/\sqrt{28})$

⇒ $\sin \angle MLN = \sin[\cos^{-1}((7\sqrt{129})/86) + \cos^{-1}(5/\sqrt{28})]$

$= \sin[\cos^{-1}((7\sqrt{129})/86)] \cos \cos^{-1}(5/\sqrt{28})$ **(note: $\cos \cos^{-1}(x) = x$)**

$+ \cos[\cos^{-1}((7\sqrt{129})/86)] \sin \cos^{-1}(5/\sqrt{28})$ **(note $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$)**

$= [(5\sqrt{43})/86][5/\sqrt{28}] + [(7\sqrt{129})/86][\sqrt{3}/\sqrt{28}]$

⇒ **$\sin \angle MLN = 23\sqrt{301}/602$**

$LN/\sin \angle LMN = MN/\sin \angle MLN$ {Law of Sines}

⇒ $LN = MN \times \sin \angle LMN/\sin \angle MLN$

⇒ $LN = (106\sqrt{3}/105) \times [7\sqrt{129})/86]/[23\sqrt{301}/602]$

⇒ **$LN = 106\sqrt{7}/115$ cm**

Area of $\Delta LMN = ½ \times LN \times MN \times \sin \angle LNM$

Area of $\Delta LMN = ½ \times (106\sqrt{7}/115) \times (106\sqrt{3}/105) \times (5/\sqrt{28})$

Area of $\Delta LMN = 2809\sqrt{3}/2415$ cm²